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DYNAMIC OPTIMIZATION WITH ASYMETRICAL PENALTIES

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Dynamic Optimization with Asymmetrical Penalties

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ABSTRACT: In this paper we discuss about dynamic programming models with a quadratic objective function. An extension is suggested to relax the hypothesis of symmetric penalties. The extended model allows for a more accurate modelling of preferences.

1. Introduction.

Mitigating the economic cycle and its undesirable consequences has been one of the main and more difficult goals of economic science. The complexity of this task is increased by two factors: 1) to implement an anticyclical policy, the decision maker must balance a variety of different and conflicting goals and 2) economic policies are usually affected by lags.

Much of the post-World War II analysis of economic fluctuations has been centred in modelling optimal policy maker's decisions by the optimization of a quadratic loss function subject to a set of constraints:

$$\text{Min} \sum_{t \in T} (x_t - x_t^d)' A_t (x_t - x_t^d) + u_t' B_t u_t \qquad (1)$$

$$\text{s.t. } \forall t \in T: x_t = f_t(x_{t-1}, u_{t-1}) \qquad (2)$$

$$x_0 = \underline{x}_0, \; u_0 = \underline{u}_0 \qquad (3)$$

where:

x_t : Vector (nx1) of state variables at time t.

x_t^d : Vector (nx1) of target values at time t.

u_t : Vector (mx1) of control variables at time t.

A_t, B_t : Symmetric semi-positive definite matrices (nxn) and (mxm).

f_t : Continuous and twice differentiable function defined from $\Re^{n+m}$ to $\Re^n$.

This family of models is quite popular because the problem is differentiable and, under standard assumptions first order conditions are linear and second order conditions can be safely ignored.

However, it has been recognized that the standard quadratic criterion is often an inaccurate representation of most policy maker's preferences. Specifically, it assumes that the preferences about misachievements are symmetrical i.e., an under-achievement in some goal is as harmful as an overachievement. This hypothesis is too restrictive in many practical situations. For example, let us suppose a macroeconomic analysis based in this model. Thus, $\mathbf{x}_t$ would include variables as inflation, GDP, unemployement rate and so on. A standard specification for $\mathbf{u}_t$ would consider some measure of the quantity of money and public expenditure. Symmetry implies that an underachievement in the GDP target, for example, is as undesirable as an overachievement. Obviously, this symmetry is an inaccurate approximation of most policy maker's preferences.

An indirect solution to this flaw have been suggested by Chow (1975) and Kendrick (1988). Friedman (1972) proposes the use of a piecewise quadratic loss function instead of **(1)** to get asymmetric penalties on deviations. This method, while correct and remarkably flexible, imposes unwieldy complexities on the optimization.

At the same time, multiobjective problems have been analyzed in other branches of optimization theory. A fruitful example of this research line is Goal Programming [Vid. Lee (1972)]. Goal Programming approaches these problems by minimizing deviations from a set of goals. These deviations are measured by means of 'soft constraints', that is, restrictions that could be violated at a certain cost.

2. The soft constrained dynamic model.

Using a 'soft constrained' model, it is easy to obtain an asymmetric penalization while keeping the problem in a linear or quadratic differentiable context. Following this approach, the asymmetric penalties model can be formalized as:

$$\mathrm{Min} \sum_{t \in T} \left(d_t^{-\prime} A_{1t} d_t^- + d_t^{+\prime} A_{2t} d_t^+ \right) + u_t' B_t u_t \tag{4}$$

$$\text{s.t.} \quad \forall t \in T: \; x_t = f_t(x_{t-1}, u_{t-1}) \tag{5}$$

$$x_t + d_t^- - d_t^+ = x_t^d \tag{6}$$

$$d_t^+, \; d_t^- \geq 0 \tag{7}$$

$$x_0 = \underline{x}_0, \; u_0 = \underline{u}_0 \tag{8}$$

where:

d_t^-: Underachievement at time t.

d_t^+: Overachievement at time t.

A_{1t}, A_{2t}: Symmetric and semi-positive definite matrices (nxn).

B_t: Symmetric and semi-positive definite matrix (mxm).

The key of this formulation lies in Eqs. (6)-(7). Eq. (6) is a 'soft constraint' in which vectors d_t^+ and d_t^- act as slack variables, allowing deviations from target at a certain (not necessarily symmetrical) cost. The nonnegativity of the slacks, as expressed in (7), is necessary in order to distinguish between surplus and deficit deviations.

There are two simplifying hypothesis that may help to solve this model:

Separability: It consists in: 1) assume the linearity of the state equation and 2) impose separability of the loss function, so matrices $\mathbf{A}_{1t}$, $\mathbf{A}_{2t}$ and $\mathbf{B}_t$ are diagonal. Thus, the problem collapses to a quadratic separable model and its solution can be computed by efficient L.P. techniques [See Wolfe (1959)] which offer substantial computational advantages over nonlinear methods.

Dimensionality reduction: When only one kind of misachievement should be penalized, the definition of unnecessary slack variables can be avoided yielding a substantial reduction of problem size. For example, let $\mathbf{I}$ be the set of goals confronted by the decision maker. This set is fully covered by:

$\mathbf{I}_u$: Set of goals in which the underachievement of target values should be penalized.

$\mathbf{I}_o$: Set of goals in which the overachievement of target values should be penalized.

If $\mathbf{I}_u \cap \mathbf{I}_o = \emptyset$, a more concise formulation of the asymmetrical problem can be stated using:

$$\forall t \in \mathbf{T}: \quad \forall i \in \mathbf{I}_u : \quad x_{it} + d_{it}^{-} \geq x_{it}^{d} \tag{9}$$

$$\forall i \in \mathbf{I}_o : \quad x_{it} - d_{it}^{+} \leq x_{it}^{d} \tag{10}$$

instead of **(6)**.

3. An example on monetary control.

We will assume that it is desired to implement an active and anticyclical monetary policy. This policy seeks to minimize the deviations of the monetary growth rate m_t from an exogenously determined target path m_t^d. In this circumstances, a policy optimization model with symmetric penalties is:

$$\text{Min} \sum_{t=0}^{T} \frac{1}{(1+r)^t} \{ \lambda(m_t - m_t^d)^2 + (1-\lambda)(m_t - m_{t-1})^2 \} \tag{11}$$

$$\text{s.t. } m_0 = \underline{m}_0 \tag{12}$$

This model implies that:

1) The decision maker is sensitive to:
   - Deviations from the target path.
   - The smoothness of the policy. This idea is captured by the quadratic adjustment cost of m_t.

   The trade-off between this two objectives is measured by the parameter λ, normalized in the simplex [0,1].

2) There is a stable trade-off (measured by a nonnegative rate of time discount r) between achieving the target in different instants.

In the asymmetrical problem:

$$\text{Min} \sum_{t=0}^{T} \frac{1}{(1+r)^t} \{ \lambda o_t^2 + (1-\lambda)(m_t - m_{t-1})^2 \} \tag{13}$$

$$\text{s.t. } m_t - o_t \geq m_t^d \tag{14}$$

$$m_0 = \underline{m}_0 \tag{15}$$

$$o_t \geq 0 \tag{16}$$

... the loss function have been modified so that only nonnegative departures from target are taken into account.

Optimal solutions for both problems have been computed by Wolfe's method [See Wolfe (1959)] using $\lambda=.4$, $r=1\%$, $\underline{m}_0=10\%$ and $T=16$. The target and optimal paths are shown in columns (C), (A) and (B) of table 1.

t	Symmetrical (A)	Asymmetrical (B)	Goal (C)	(A)-(C)	(B)-(C)
1	8.37	6.36	2.91	5.46	3.45
2	6.82	5.00	4.64	2.18	0.36
3	5.28	3.86	5.00	0.28	-1.14
4	3.93	2.72	4.14	-0.21	-1.42
5	2.41	1.56	2.48	-0.07	-0.92
6	0.83	0.39	0.53	0.30	-0.14
7	-0.56	-0.79	-1.20	0.64	0.41
8	-1.54	-1.71	-2.36	0.82	0.65
9	-1.96	-2.20	-2.77	0.81	0.57
10	-1.85	-2.31	-2.47	0.62	0.16
11	-1.33	-2.30	-1.65	0.32	-0.65
12	-0.58	-2.30	-0.59	0.01	-1.71
13	0.18	-2.30	0.42	-0.24	-2.72
14	0.78	-2.30	1.15	-0.37	-3.45
15	1.14	-2.30	1.50	-0.36	-3.80
16	1.26	-2.30	1.44	-0.18	-3.74

Table 1: Symmetrical vs. asymmetrical control results.

Solution (A) may be formally optimal, but could never be satisfactory if preferences about misachievements are asymmetric. Specifically, the path (A) tends to grow toward the goal path for $t \geq 10$, while path (B) reaches a steady state. This equilibrium will be altered only if the target shifts under the optimal path, yielding overachievements once again. This effect is illustrated in fig. 1.

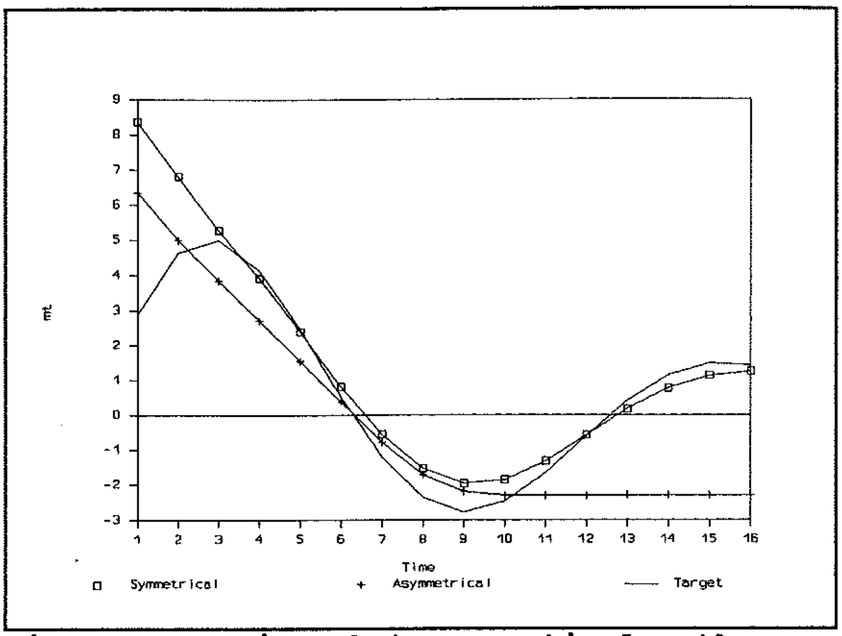

Fig 1: Comparison between optimal paths.

It can also be seen that path (B) is more effective because the surroundings of the target path are reached in t=2, while the symmetrical penalties path does not achieve a similar proximity up to t=3.

4. Conclusions.

We have shown that dynamic problems with asymmetric penalties can be easily formulated using multiobjective programming techniques so:

a) The inaccurate modelling of asymmetric preferences is avoided.

b) The model is kept in a linear or quadratic differentiable context.

We have presented all the formulations in a deterministic framework in order to avoid unnecessary complexities. The stochastic extension of this technique is straightforward.

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